One-point velocity statistics in decaying homogeneous isotropic turbulence

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A reasonable closure of the Monin-Lundgren hierarchy of equations for many-point probability density functions of velocity for decaying homogeneous isotropic turbulence is achieved to treat the first equation for the one-point velocity distribution. As a result, we are naturally led to a two-parameter family of solutions for the distribution, which are Gaussian.

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I. INTRODUCTION

The so-called Monin-Lundgren (ML) hierarchy of equations for many-point probability density functions (PDF) of velocity in wall-free incompressible turbulence was presented in 1967 independently by Monin [1] and Lundgren [2]. It looked simpler but less perfect than the other statistical (functional) formalism of turbulence by Hopf [3]. But it was recently proved [4] that these two statistical formalisms are mathematically equivalent if the hierarchy continues up to infinity. Both formalisms have looked mathematically so formidable to treat that few people have been much interested in them. In this context, it may be noted that Lewis and Kraichnan [5] reconstituted the Hopf formalism extensively using solenoidal velocity field with the space-time argument, and Hosokawa [6] further generalized this so as to treat arbitrary stochastic fields with the space-time argument and introduced a technique of functional integration to give a (formal) general solution of the functional basic equation; next based on this formalism, statistical hydromechanics with general random-force action was formulated in [7], including the Novikov equation [8] as a special case. The Hopf functional, too, embraces in itself the equations for all velocity correlation tensors chained to make an infinite hierarchy. Many kinds of (low-order) closure approach to this hierarchy have been hitherto tried mostly to analyze homogeneous isotropic turbulence, in particular, in an attempt to derive the energy spectrum of Kolmogorov and the Kolmogorov constant. But they are not touched here but well summarized, for example, in the book of McComb [9], who categorized some of them as renormalized perturbation theory. The one-point velocity statistics, however, has never been a target from this research sight.

Recently, however, the author found that, if the coincidence condition and Kolmogorov's formulas of variance of velocity difference between two points (for the viscous range) are combined in a reasonable way in the viscous term of the first equation in the Monin-Lundgren hierarchy, the two-point PDF therein can be transformed to the one-point PDF times a certain factor so that the first equation may be closed under the condition of homogeneity. Fortunately, the resulting equation is easy to solve. The result gives a two-parameter family of Gaussian similarity solutions (including the power law of energy decay), which gives an answer to the deep question of why the one-point PDF of velocity is so close to Gaussian in decaying isotropic turbulence, since Batchelor [10]. The present process for this issue does not need an appeal to the central limit theorem, nor to a particular mechanism of random force action.

However, after the theoretical research for the sub-Gaussianity of the one-point PDF in forced turbulence by Falkovich and Lebedev [11] about a decade ago, people may tend to believe that the tails of the one-point PDF in every turbulence decreases more rapidly than Gaussian. Their theory is heavily based on forcing action (and even the degree of sub-Gaussianity depends on a particular form of the PDF of random force), while the present Monin-Lundgren hierarchy has no such forcing term. There were some experiments and direct numerical simulations (DNSs) to support the sub-Gaussianity, but all of them treated stationary forced isotropic turbulence, to the author's knowledge. In contrast, the experiment by Makita [12] that treats the decaying homogeneous turbulence behind an active grid (achieving Taylor-scale Reynolds number $R_\lambda \approx 387$) is unlikely to evidence such a sub-Gaussianity. The DNSs by Yamamoto and Kambe [13] and Oide, Hosokawa, and Yamamoto [14] which deal with decaying isotropic turbulence at $R_\lambda \approx 100$ and 160, respectively, do not indicate such an apparent deviation from Gaussianity, either, as that by Vincent and Meneguzzi [15] who treated forced turbulence at $R_\lambda \approx 150$. In view of these facts, it still looks too early to extend the sub-Gaussianity argument upon every isotropic turbulence. Jun *et al.*'s experiment [16] which treated both forced and decaying two-dimensional turbulence shows even a super-Gaussianity in the PDF tails for the latter case, although this case must be largely affected by the restriction peculiar to two-dimensionality. Anyhow, it is natural to think that there may be a delicate but fundamental difference in the one-point PDF of velocity (or large-scale nature of velocity field) between forced turbulence and decaying turbulence, because the one is always randomly disturbed by energy supply into fixed large-scale components of velocity field to sustain a stationary state while the other is not so but just decays keeping an autonomous energy-cascading turbulent structure.

In Sec. II, the first equation of the hierarchy is shown, in which the one-and two-point PDFs of velociy are involved consistently with the Navier-Stokes equation for incompressible flow. There we shall see how simple it is if the conditions of homogeniety and isotropy are taken into account. In Sec. III a special consideration of the coincidence condition and Kolmogorov's formula of variance of velocity difference between two points (for the viscous range) will play a decisive role of reducing the two-point PDF left in the equation to the one-point PDF times an unknown factor which involves the time-dependent dissipation function. Hence we have formally a closed partial differential equation for the one-point PDF, the solution of which is shown in Sec. IV to be generally Gaussian, whatever form the dissipation function may take. It is out of scope here to pursue all possible forms of the dissipation function, but for a two-parameter family of exact solutions which are found in Sec. V. So it is clarified that with these solutions, the Monin-Lundgren hierarchy is truncated at the first equation, and the two parameters can characterize the total hiererarchy of equations.

II. THE MONIN-LUNDGREN HIERARCHY FOR HOMOGENEOUS TURBULENCE

We write the s-point PDF of velocity as $F_s(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_s,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_s,t)$, where $\boldsymbol{v}$ is the velocity vector of the flow at the position $\boldsymbol{r}$ and t is the time variable. Then the Monin-Lundgren hierarchy for a wall-free incompressible turbulence may be written as

$$\frac{\partial F_s}{\partial t} = -\sum_{i=1}^{s} \boldsymbol{v}_i \cdot \frac{\partial F_s}{\partial \boldsymbol{r}_i} + \sum_{i=1}^{s} \frac{\partial}{\partial \boldsymbol{v}_i} \cdot \left\{ \frac{1}{4\pi} \int\int \frac{\partial}{\partial \boldsymbol{r}_i} \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_{s+1}|} \left(\boldsymbol{v}_{s+1} \cdot \frac{\partial}{\partial \boldsymbol{r}_{s+1}} \right)^2 \times F_{s+1} d\boldsymbol{v}_{s+1} d\boldsymbol{r}_{s+1} - \lim_{\boldsymbol{r}_{s+1}\to \boldsymbol{r}_i} \nu \left(\frac{\partial}{\partial \boldsymbol{r}_{s+1}} \cdot \frac{\partial}{\partial \boldsymbol{r}_{s+1}} \right) \int \boldsymbol{v}_{s+1} F_{s+1} d\boldsymbol{v}_{s+1} \right\} \quad (1)$$

for $s \geqslant 1$ [4]. ν is the kinematic viscosity.

In order for F_s to be the s-point velocity distribution, there are some conditions to be obeyed, as follows.

Reduction condition:

$$\int F_{s+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{s+1},\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{s+1},t) d\boldsymbol{v}_{s+1} = F_s(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_s,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_s,t). \quad (2)$$

Needless to say, the right-hand side has no argument of $\boldsymbol{r}_{s+1}$ by definition of the joint PDF F_s. When $s=0$, the right-hand side is unity by definition. This is nothing but the normalization condition of the one-point PDF.

Coincidence condition:

$$\int F_{s+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{s+1},\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{s+1},t)\,\delta(\boldsymbol{r}_s - \boldsymbol{r}_{s+1}) d\boldsymbol{r}_{s+1} = F_s(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_s,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_s,t)\,\delta(\boldsymbol{v}_s - \boldsymbol{v}_{s+1}). \quad (3)$$

Separation condition: When some points are very far apart from others, the distribution functions for them become independent of each other. If s points are apart this way, we must have

$$F_s(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_s,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_s,t) = F_1(\boldsymbol{v}_1,\boldsymbol{r}_1,t)\cdots F_1(\boldsymbol{v}_s,\boldsymbol{r}_s,t). \quad (4)$$

Divergence condition: Since we treat an incompressible flow, the average velocity is divergence-free at any point,

$$\frac{\partial}{\partial \boldsymbol{r}_i} \cdot \int \boldsymbol{v}_i F_s(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_s,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_s,t) d\boldsymbol{v}_i = 0 \quad (5)$$

for $1 \leqslant i \leqslant s$.

So far, approximative treatments of decaying homogeneous isotropic turbulence based on this Monin-Lundgren hierarchy were executed by Ulinich and Lyubimov [17], Tatsumi and Yoshimura [18], and Hosokawa [19]. All these works predict Gaussian forms of $F_1(\boldsymbol{v}_1,t)$ (which must be independent of $\boldsymbol{r}_1$ because of homogeneity) and energy decay laws with some power indices. A much simpler method is reported here.

For $s=1$, Eq. (1) becomes

$$\frac{\partial F_1}{\partial t} = -\boldsymbol{v}_1 \cdot \frac{\partial F_1}{\partial \boldsymbol{r}_1} + \frac{\partial}{\partial \boldsymbol{v}_1} \cdot \left\{ \frac{1}{4\pi} \frac{\partial}{\partial \boldsymbol{r}_1} \int\int \frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \left(\boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{r}_2} \right)^2 \times F_2 d\boldsymbol{v}_2 d\boldsymbol{r}_2 - \lim_{\boldsymbol{r}_2 \to \boldsymbol{r}_1} \nu \left(\frac{\partial}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{r}_2} \right) \int \boldsymbol{v}_2 F_2 d\boldsymbol{v}_2 \right\}. \quad (6)$$

If the turbulence is homogeneous, F_1 must be independent of $\boldsymbol{r}_1$. Therefore the first term on the right-hand side of Eq. (6) vanishes. On the other hand, F_2 should not depend on two positions separately but only on the difference: $r=|\boldsymbol{r}|=|\boldsymbol{r}_2-\boldsymbol{r}_1|$ because of homogeneity and isotropy. Then, the double integral inside the curly brackets cannot depend on $\boldsymbol{r}_1$, so that the gradient operation with respect to $\boldsymbol{r}_1$ makes the whole term vanish. Thus the first equation of the hierarchy (1) is simply reduced to

$$\frac{\partial F_1(\boldsymbol{v}_1,t)}{\partial t} = -\lim_{\boldsymbol{r}_2 \to \boldsymbol{r}_1} \nu \left(\frac{\partial}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{r}_2} \right) \frac{\partial}{\partial \boldsymbol{v}_1} \cdot \int \boldsymbol{v}_2 F_2(\boldsymbol{v}_1,\boldsymbol{v}_2,r,t) d\boldsymbol{v}_2. \quad (7)$$

III. THE FORM OF F_2 IN THE VISCOUS RANGE OF r

Let us notice here that the knowledge of the F_2 on the right-hand side of Eq. (7) which is necessary to solve this is only how it behaves in the viscous range of r around $r=0$, as is obvious from the limit operator restricting $\boldsymbol{r}_2$. We now determine the proper (for isotropic turbulence) local form of F_2 compatible with the coincidence condition.

First, we start from the assumption that F_2 can be expressed as $g_+[(\boldsymbol{v}_2+\boldsymbol{v}_1)/2,r,t]g_-(\boldsymbol{v}_2-\boldsymbol{v}_1,r,t)$ for very small r. This means that the velocity difference and sum are statistically independent of each other there. Then, the coincidence condition (3) for $s=1$:

$$\lim_{\boldsymbol{r}_2 \to \boldsymbol{r}_1} F_2 = F_1 \delta(\boldsymbol{v}_2 - \boldsymbol{v}_1), \quad (8)$$

requires $g_+[(\boldsymbol{v}_2+\boldsymbol{v}_1)/2,0,t]=F_1[(\boldsymbol{v}_2+\boldsymbol{v}_1)/2,t]$ and $g_-(\boldsymbol{v}_2-\boldsymbol{v}_1,0,t)=\delta(\boldsymbol{v}_2-\boldsymbol{v}_1)$. Therefore for r slightly away from zero it is natural to add the linear terms in the Taylor expansion around $r=0$ to these functions, only if they are analytic in r. But such a term cannot exist for g_+ since F_1 is independent of r (by homogeneity), nor for g_- because the delta function is singular. However, as for g_- we can consider the possibility of using an approximate expression of the delta function with a parameter n which is going to infinity: $\delta_n(\boldsymbol{v}_2-\boldsymbol{v}_1)$ as a $g_-(\boldsymbol{v}_2-\boldsymbol{v}_1,1/n,t)$ (see Ref. [20]). For our three-dimensional case, we define

$$\delta_{1/x}(v_{2x}-v_{1x})\delta_{1/y}(v_{2y}-v_{1z})\delta_{1/z}(v_{2z}-v_{1z}) \equiv \delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1),$$

where, of course, $\boldsymbol{r}=(x,y,z)$. Among some candidates for $\delta_{1/\boldsymbol{r}}$, we have to choose the most suitable one, as follows.

It is necessary that all the three component functions of $\delta_{1/\boldsymbol{r}}$ are well-behaved and analytic in $\boldsymbol{r}>0$, and further it is essential that they have the definite variances which are in accord with Kolmogorov's deduction [21] in the viscous range of $\boldsymbol{r}$ in isotropic turbulence; that is, in our three-dimensional frame of velocity space, expressed as

$$\langle(v_{2x}-v_{1x})^2\rangle = \sigma_1^2x^2+\sigma_2^2y^2+\sigma_2^2z^2 \equiv V_x,$$

$$\langle(v_{2y}-v_{1y})^2\rangle = \sigma_2^2x^2+\sigma_1^2y^2+\sigma_2^2z^2 \equiv V_y,$$

$$\langle(v_{2z}-v_{1z})^2\rangle = \sigma_2^2x^2+\sigma_2^2y^2+\sigma_1^2z^2 \equiv V_z, \tag{9}$$

with $\sigma_1^2=\frac{\varepsilon}{15\nu}$, and $\sigma_2^2=\frac{2\varepsilon}{15\nu}$ (See also [22]). Here $\varepsilon\equiv\varepsilon(t)$ is the (time-dependent) average energy dissipation rate per unit mass in decaying homogeneous isotropic turbulence and an angular bracket means the ensemble average. To understand Eq. (9), just recall that the x component of velocity difference is longitudinal to the x axis, but transverse to the y and z axes, and the like. [Then, for example, if $\boldsymbol{r}_2-\boldsymbol{r}_1=(x,0,0)$, we have the right answer: $\langle(v_{2x}-v_{1x})^2\rangle=\sigma_1^2x^2$, $\langle(v_{2y}-v_{1y})^2\rangle=\sigma_2^2x^2$, $\langle(v_{2z}-v_{1z})^2\rangle=\sigma_2^2x^2$.] Obviously, the *well-behaved* and *analytic* form of $\delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1)$ which satisfies Eq. (9) must be a *three-dimensional (normalized) Gaussian function precisely with these variances*, then each component of which is

$$\frac{1}{\sqrt{2\pi V_x}}\exp\left[-\frac{(v_{2x}-v_{1x})^2}{2V_x}\right]$$

and the like. Accordingly, it is reasonable to redefine $\delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1)$ precisely as

$$\delta_{1/\sqrt{2V_x}}(v_{2x}-v_{1x})\delta_{1/\sqrt{2V_y}}(v_{2y}-v_{1z})\delta_{1/\sqrt{2V_z}}(v_{2z}-v_{1z}) \equiv \delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1).$$

Thus we have no choice but to express the F_2 for the viscous range of r in Eq. (7) by

$$F_2=F_1\left(\frac{\boldsymbol{v}_2+\boldsymbol{v}_1}{2},t\right)\delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1). \tag{10}$$

Indeed, this form behaves well for $r>0$ as it should in the viscous range and it exactly satisfies the coincidence condition in the limit $r=0$; only when $\delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1)=\delta(\boldsymbol{v}_2-\boldsymbol{v}_1)$. [Note in addition that when we put $\boldsymbol{u}_+=(\boldsymbol{v}_2+\boldsymbol{v}_1)/2$ and $\boldsymbol{u}_-=\boldsymbol{v}_2-\boldsymbol{v}_1$, the Jacobian caused by this change of variables is just unity.] After all, we may say that this form for very small r is universal for the locally isotropic turbulence (treated by Kolmogorov), because all possible non-Gaussian deflections which may appear in any higher-order than the second-order moment of $(\boldsymbol{v}_2-\boldsymbol{v}_1)$ should be negligible for r near zero. [Imagine the Gram-Charlier series associated with the Gaussian function $\delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1)$ now determined.]

Here we Taylor-expand F_1 in Eq. (10) around $\boldsymbol{v}=\boldsymbol{v}_1$ as

$$F_1(\boldsymbol{v}_1,t)+\frac{(\boldsymbol{v}_2-\boldsymbol{v}_1)}{2}\cdot\frac{\partial}{\partial\boldsymbol{v}_1}F_1(\boldsymbol{v}_1,t),$$

neglecting the higher-order terms. Then, we find that the first term with $F_1(\boldsymbol{v}_1,t)$ when inserted in the integral of Eq. (7) is integrated to become just $\boldsymbol{v}_1F_1(\boldsymbol{v}_1,t)$, which should vanish in Eq. (7) since it is independent of $\boldsymbol{r}_2$. The second term with the derivative of $F_1(\boldsymbol{v}_1,t)$ times $(\boldsymbol{v}_2-\boldsymbol{v}_1)$ decisively contributes to the integration in cooperation with $\boldsymbol{v}_2$ $[=(\boldsymbol{v}_2-\boldsymbol{v}_1)+\boldsymbol{v}_1]$ and $\delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1)$, eventually to bring forth the form

$$\begin{aligned}\frac{\partial F_1(\boldsymbol{v}_1,t)}{\partial t} &= -\lim_{\boldsymbol{r}\to\infty}\nu\left(\frac{\partial}{\partial\boldsymbol{r}}\cdot\frac{\partial}{\partial\boldsymbol{r}}\right)\sum_{i=x,y,z}V_i\frac{\partial}{\partial v_{1i}}\frac{\partial}{\partial v_{1i}}F_1(\boldsymbol{v}_1,t)/2\\ &= -\nu(2\sigma_1+4\sigma_2)\frac{\partial}{\partial\boldsymbol{v}_1}\cdot\frac{\partial}{\partial\boldsymbol{v}_1}F_1(\boldsymbol{v}_1,t)/2\\ &= -\frac{\varepsilon(t)}{3}\frac{\partial}{\partial\boldsymbol{v}_1}\cdot\frac{\partial}{\partial\boldsymbol{v}_1}F_1(\boldsymbol{v}_1,t),\end{aligned} \tag{11}$$

using relations (9) carefully. Note that this holds for any value of $\nu>0$ (but small enough to keep turbulence). This equation seems to be a simple closed equation for $F_1(\boldsymbol{v}_1,t)$ but it might not be the case. $\varepsilon(t)$ (>0) could be essentially interrelated with higher-order than the first equations. Therefore it is generally unknown at this stage [until it is fixed in Eq. (19)].

We can justify Eq. (11) from another angle of view. Let us multiply $\int d\boldsymbol{v}_1\boldsymbol{v}_1^2/2$ to both sides of the equation. Then we have

$$\begin{aligned}-\varepsilon(t) &= -\frac{\varepsilon(t)}{3}\int\boldsymbol{v}_1^2/2\frac{\partial}{\partial\boldsymbol{v}_1}\cdot\frac{\partial}{\partial\boldsymbol{v}_1}F_1(\boldsymbol{v}_1,t)d\boldsymbol{v}_1\\ &= -\frac{\varepsilon(t)}{3}4\pi\int\frac{v_1^4}{2}\frac{\partial}{v_1^2\partial v_1}v_1^2\frac{\partial}{\partial v_1}F_1(v_1,t)dv_1.\end{aligned} \tag{12}$$

It is easy to prove that both sides are identical for any $\varepsilon(t)$, since there must be the normalization condition:

$$\int F_1(\boldsymbol{v}_1,t)d\boldsymbol{v}_1=1. \tag{13}$$

This important fact insures self-consistency of Eq. (11) that has led to this. A better theory on our problem to replace Eq. (11), if any, should not be free from this consistency test.

Ulinich and Lyubimov [17] obtained Eq. (11) by their unique approach to the Monin-Lundgren hierarchy using the $\mathrm{Re}^{-1/4}$ expansion (Re: Reynolds number), as the first approximation, together with two other equations which lead to solving for $\varepsilon(t)$ simultaneously. Tatsumi and Yoshimura [18] obtained Eq. (11) by using their cross-independence hypothesis that boldly assumes independence from each other of the PDFs of velocity difference and sum between two arbitrary points in turbulence, and determined a form of $\varepsilon(t)$ by a method of similarity solution. But this hypothesis embraces a serious defect when applied to the inertial range of r (see [19,23]).

We note that there have, so far, been two other model equations dealing with the one-point velocity statistics which aim at wider application to various turbulent flows. First, Lundgren [24] reformed the first Eq. (6) quite drastically, by adopting for the random pressure term the concept of relaxation of the PDF of random velocity to the locally isotropic Gaussian form during relaxation time (by analogy with the Bhatnager, Gross, and Krook model in gas kinetics [25]) and for the random viscous term an intuitive form which is proportional to the difference of random velocity and its local mean, in the way that dissipation rate and kinetic energy are involved in the two terms. Therefore this is nothing but a model, perhaps useful for some flows he took up in his paper, but it is no miracle that his model does not necessarily lead us to Eq. (11) for homogeneous isotropic turbulence; curiously his solution for this case is not guaranteed to be Gaussian unless initially it is so, though with an arbitrarily decaying variance.

Second, Pope's model equation for the one-point velocity [26] is different from Lundgren's only at one point that both the random pressure term and the random viscous term have been replaced by the terms to be brought from a Langevin dynamics:

$$dv_i = \sqrt{C_0 \varepsilon} dW_i + G_{ij}(v_j - \langle U_j \rangle) dt,$$

where C_0 is called Kolmogorov's constant in his paper, dW_i is an isotropic Wiener process, G_{ij} is a tensor function of local mean quantities, and repetitive suffixes obey the summation convention. Here, the Markovian random force with dW_i causes the random pressure in contrast to Lundgren's idea and brings forth the term with a second-order derivative of the PDF of velocity, just as $\frac{1}{2}C_0\varepsilon\frac{\partial^2 F_1}{\partial v_i \partial v_i}$, into the equation for F_1 so that it becomes a typical Fokker-Planck equation. The next friction-type force with G_{ij} adds the random viscous term due to a similar idea to that of Lundgren into the equation for F_1, but G_{ij} is far more complex depending on types of flow; but, in order to be physically meaningful, this tensor would be negative definite. Thus it is clear that Pope's model equation (even for homogeneous isotropic case) is fundamentally distinct from Eq. (11). The former is obviously a Fokker-Planck equation which is usually ensured to have a final steady solution, while the latter is not such because it does not have a *positive-definite* diffusion term. The steady Gaussian solution we can expect from Pope's equation for the homogeneous isotropic case is, only if we assume that $G_{ij} = -\gamma\delta_{ij}$, $F_1 \sim \exp(-\frac{\gamma}{C_0\varepsilon} v_i v_i)$. Apparently, this expresses a stationary turbulence sustained by (suppositional) energy injected by the Markovian random force, but not a decaying turbulence. This situation would be changed by giving ε time-dependence in a self-consistent way somehow.

IV. GAUSSIAN PDF OF THE ONE-POINT VELOCITY

By assuming that $\varepsilon(t)$ is known and transforming t into $\tau > 0$ as

$$\tau = -\int^t \varepsilon(t) dt, \tag{14}$$

Eq. (11) becomes *formally* a simple three-dimensional diffusion equation on the τ axis:

$$\frac{\partial F_1}{\partial \tau} = \frac{1}{3} \frac{\partial}{\partial \boldsymbol{v}_1} \cdot \frac{\partial}{\partial \boldsymbol{v}_1} F_1. \tag{15}$$

Therefore $F_1(\boldsymbol{v}_1, \tau)$ can be solved as

$$F_1(\boldsymbol{v}_1, \tau) = \frac{1}{(2\pi \cdot 2\tau/3)^{3/2}} \exp\left[-\frac{\boldsymbol{v}_1^2}{4\tau/3} \right]. \tag{16}$$

Thus the PDF of velocity at one point should be Gaussian and isotropic in general, whatever the functional form of $\tau(t)$ (>0) may be. Obviously $\tau(t)$ means kinetic energy per unit mass, and it must decay with t, as Eq. (14) indicates: $d\tau/dt = -\varepsilon(t)$. And at the same time the PDF of velocity tends to concentrate to the delta function, $\delta(\boldsymbol{v}_1)$, at $\tau = 0$, so that it is physically natural to assign the lower limit of the integral in Eq. (14) as $t = \infty$.

Although it is out of scope here to search all the forms of $\tau(t)$, the present situation around this issue is as follows. Ulinich and Lyubimov [17] gave

$$\tau(t)/\tau(0) = (1 + t/t_0)^{-k} \tag{17}$$

in their approximation (k: const). Tatsumi and Yoshimura [18] obtained $\tau(t) \sim t^{-1}$. Since the theoretical predictions of $\tau(t) \sim t^{-10/7}$ (Kolmogorov [21]) and $\tau(t) \sim t^{-6/5}$ (Saffman [27]), various values of the power law index of energy decay in homogeneous isotropic turbulence have been reported from experimental and theoretical sides, but the problem is still open [28,29]. The power index seems to be related with the energy spectrum or how the turbulence was created in the past.

V. TWO-PARAMETER FAMILY OF EXACT SPECIAL SOLUTIONS FOR $\tau(t)$

Let us introduce the similarity variable $\boldsymbol{w} = \boldsymbol{v}_1 t^\alpha$, and write $F_1(\boldsymbol{v}_1, t) = t^{3\alpha}\Phi(\boldsymbol{w})$. Then we have in place of Eq. (11)

$$3\alpha\Phi + \alpha \boldsymbol{w} \cdot \frac{\partial \Phi}{\partial \boldsymbol{w}} = -\frac{\varepsilon(t)}{3} t^{2\alpha+1} \frac{\partial}{\partial \boldsymbol{w}} \cdot \frac{\partial \Phi}{\partial \boldsymbol{w}}. \tag{18}$$

Since t and $\boldsymbol{w}$ are independent of each other, in order for this equation to be meaningful we should have

$$\frac{\varepsilon(t)}{3} t^{2\alpha+1} \equiv \beta = \text{const}(>0) \text{ so that } \varepsilon(t) = 3\beta/t^{2\alpha+1}. \tag{19}$$

Thereupon it is easy to obtain the normalized solution for Φ:

$$\Phi(\boldsymbol{w}) = \frac{1}{(2\pi\beta/\alpha)^{3/2}} \exp\left(-\frac{\alpha \boldsymbol{w} \cdot \boldsymbol{w}}{2\beta} \right) \tag{20}$$

where $\alpha > 0$ is necessary and, from relations (14) and (19),

$$\tau(t) = \frac{3\beta}{2\alpha} \Big/ t^{2\alpha}, \tag{21}$$

which satisfies the condition: $\tau(\infty)=0$. Needless to say, this identifies the power law of energy decay and the parameter 2α indicates the power law index, while β is determined by α and the kinetic energy per unit mass at a certain specific time. From solution (20), we have a two-parameter family of exact solutions for the one-point velocity distribution as

$$F_1(\boldsymbol{v}_1,t) = \frac{t^{3\alpha}}{(2\pi\beta/\alpha)^{3/2}} \exp\left(-\frac{\alpha t^{2\alpha}\boldsymbol{v}_1\cdot\boldsymbol{v}_1}{2\beta}\right). \tag{22}$$

[This solution is based on the similarity method starting from Eq. (18), but there might be other solutions not belonging to this family.]

The main character of the turbulence obtained here is as follows. Solution (22) gives the root mean square of one component of velocity as

$$u(t) \equiv (\beta/\alpha)^{1/2} t^{-\alpha}; \tag{23}$$

a large-scale length is defined as

$$L(t) \equiv u(t)^3/\varepsilon(t) = \frac{\beta^{1/2}}{3\alpha^{3/2}} t^{-\alpha+1}; \tag{24}$$

the Kolmogorov length is

$$\eta(t) = [\nu^3/\varepsilon(t)]^{1/4} = \left(\frac{\nu^3}{3\beta}\right)^{1/4} t^{(2\alpha+1)/4}; \tag{25}$$

and a large-scale Reynolds number can be defined as

$$\mathrm{Re}(t) \equiv u(t)L(t)/\nu = \frac{\beta}{3\nu\alpha^2} t^{-2\alpha+1} = \left(\frac{L(t)}{\eta(t)}\right)^{4/3}. \tag{26}$$

The Taylor microscale defined as $\lambda = L\sqrt{15/\mathrm{Re}}$ is

$$\lambda(t) = \sqrt{\frac{5\nu}{\alpha}} t^{1/2}. \tag{27}$$

The time dependence of only this quantity is, notably, irrespective of α.

The case of $\alpha=1/2$ in the family is the same as what was obtained by Tatsumi and Yoshimura [18]. But this turbulence has a hardly acceptable nature that the Reynolds number (26) is time-invariant. Obviously, the cases with $\alpha > 1/2$ will better explain the mostly observed aspects of real decaying homogeneous isotropic turbulence [28]. What can be said from the present consideration is that solutions are not unique but diversified in the two-parameter family. Such a diversity must originate in a mechanism by which turbulence is created. Exactly speaking, the two parameters determine the intensity of turbulence represented by $u(t)$ (23) as well as the large-scale length $L(t)$ (24) together with their time dependence in a unique sense. A two-parameter family of solutions were previously inferred by the author in a crude way [19]. Interestingly, this family (with parameters α_0 and a) are in complete accord with the present family when we put $\alpha_0 = \beta$ and $a=(1+2\alpha)/4$.

VI. CONCLUSION

We have truncated the Monin-Lundgren hierachy for many-point PDFs of velocity in decaying homogeneous isotropic turbulence to obtain the closed-form first equation for $F_1(\boldsymbol{v}_1,t)$, by introducing the reasonable Gaussian function $\delta_{1/\boldsymbol{r}}(\boldsymbol{v}_2-\boldsymbol{v}_1)$ to simplify the form of F_2 for the viscous range of $\boldsymbol{r}$, and found a two-parameter family of exact solutions for this equation, which are all Gaussian. Then, the possibility to find subGaussian solutions of Eq. (11) or originally Eq. (7) would be rare. Therefore the Gaussianity would still remain as a typical candidate to characterize the PDF of velocity in decaying homogeneous isotropic turbulence, from a theoretical viewpoint apart from unavoidable experimental or numerical error. In order to know the structural knowledge of turbulence such as autocorrelations of velocity, energy spectrum, and the PDF of velocity increment, however, it is necessary to solve the second equation of the Monin-Lundgren hierarchy even approximately [19]. This is a significant task mostly left for the future. It may be noticed that the present trick using $\delta_{1/\boldsymbol{r}}$ would be useful for the viscous terms at any stage of the hierarchy (1), which involves F_{s+1}, to be simplified like $-\frac{\varepsilon(t)}{3}\frac{\partial}{\partial \boldsymbol{v}_j}\cdot\frac{\partial}{\partial \boldsymbol{v}_j}F_s(\ldots,\boldsymbol{v}_j,\ldots)$ [with a known $\varepsilon(t)$ as Eq. (19) if the turbulence belongs to the two-parameter family; therefore a direct relationship of the two parameters with the fine structure of isotropic turbulence is expected to be found by solving the second equation for F_2].

Finally, it would be worth noting that for two extreme cases of isotropic turbulence with Reynolds number$=\infty$ and 0, there are exact solutions of the Hopf (characteristic functional) equation which insure the Gaussianity of velocity field. The first one is the *inviscid* steady Gaussian functional with the white energy spectrum found by Hopf himself [3], although this solution is unrealistic. The second one is a Gaussian functional corresponding to the final period of decay, when we deal with the linearized Navier-Stokes equation: $\partial \boldsymbol{u}/\partial t = \nu\Delta \boldsymbol{u}$ (Δ: Laplacian). See the Appendix below for the details of this case.

APPENDIX

By definition, the Hopf functional is

$$\Psi[\boldsymbol{y}(\boldsymbol{r}),t] = \int \exp\left[i\int d\boldsymbol{r}\boldsymbol{y}(\boldsymbol{r})\cdot T^{t-t_0}\boldsymbol{u}(\boldsymbol{r})\right] P[\boldsymbol{u}(\boldsymbol{r}),t_0]\delta\boldsymbol{u}, \tag{A1}$$

where T^{t-t_0} is the evolutional operator from time t_0 to t, that may be replaced by the *linear* operator $e^{\nu(t-t_0)\Delta}$ for this case. $P[\boldsymbol{u}(\boldsymbol{r}),t_0]\delta\boldsymbol{u}$ denotes the differential probability of $\boldsymbol{u}(\boldsymbol{r})$ at $t=t_0$, and then the outside integral means a functional integration [30,31] over the function space of $\boldsymbol{u}(\boldsymbol{r})$. Equation (A1) is equivalent to the Hopf equation, since it is derived by time differentiation of Eq. (A1).

On the other hand, we have

$$P[\boldsymbol{u}(\boldsymbol{r}),t] = \int \exp\left[-i\int d\boldsymbol{r}\boldsymbol{y}(\boldsymbol{r})\cdot\boldsymbol{u}(\boldsymbol{r})\right]\Psi[\boldsymbol{y}(\boldsymbol{r}),t]\delta\boldsymbol{y} \tag{A2}$$

which is a functional inverse Fourier transform. If Eqs. (A1) and (A2) are combined,

$$\Psi[\boldsymbol{y}(\boldsymbol{r}),t] = \int\int \exp\left\{ i \int d\boldsymbol{r}[\boldsymbol{y}(\boldsymbol{r})T^{t-t_0} - \boldsymbol{y}'(\boldsymbol{r})]\cdot \boldsymbol{u}(\boldsymbol{r}) \right\} \times \Psi[\boldsymbol{y}'(\boldsymbol{r}),t_0]\delta\boldsymbol{y}'\,\delta\boldsymbol{u} \quad \text{(A3)}$$

is obtained.

Let us assume that the initial Hopf functional is homogeneous isotropic and Gaussian as

$$\Psi[\boldsymbol{y}'(\boldsymbol{r}),t_0] = \exp\left[- \int\int d\boldsymbol{r}d\boldsymbol{r}'\boldsymbol{y}'(\boldsymbol{r})\boldsymbol{y}'(\boldsymbol{r}'):\boldsymbol{Q}(\boldsymbol{r}-\boldsymbol{r}') \right], \quad \text{(A4)}$$

where $\boldsymbol{Q}(\boldsymbol{r}-\boldsymbol{r}')$ denotes the correlation tensor of velocity field. Since the functional integration with respect to $\boldsymbol{u}$ in Eq. (A3) gives rise to a delta functional [30], $\delta[\boldsymbol{y}(\boldsymbol{r})T^{t-t_0}-\boldsymbol{y}'(\boldsymbol{r})]$, insertion of Eq. (A4) into Eq. (A3) leads to

$$\Psi[\boldsymbol{y}(\boldsymbol{r}),t] = \exp\left[- \int\int d\boldsymbol{r}d\boldsymbol{r}'\boldsymbol{y}(\boldsymbol{r})\boldsymbol{y}(\boldsymbol{r}'):e^{\nu(t-t_0)(\Delta+\Delta')}\boldsymbol{Q}(\boldsymbol{r}-\boldsymbol{r}') \right], \quad \text{(A5)}$$

where the primed Laplacian operates to $\boldsymbol{r}'$.

Although this result is very symbolic, it is apparent that the Gaussianity of velocity field is kept long until energy totally decays. When treated in the wave-number space rather than the physical space, everything would be easier to grasp; we shall see the energy spectrum in place of $\boldsymbol{Q}(\boldsymbol{r}-\boldsymbol{r}')$. Anyway it is to be noted that the Gaussianity comes *not from the central limit theorem but the initial condition for the Hopf functional* in this case with a *linear* dynamics, in contrast to Batchelor's viewpoint about this problem [10]. The reason to choose the Gaussian form initially may be given by the (information) entropy-functional maximum principle under the condition of a given $\boldsymbol{Q}(\boldsymbol{r}-\boldsymbol{r}')$ [31].

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